

ON SOME ASYMPTOTIC FORMULAS FOR CURVES IN POSITIVE CHARACTERISTIC

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ABSTRACT. We prove some general asymptotic formula for the values of L -function of a sequence of constructible \mathbb{Q}_l -sheaves on curves over \mathbb{F}_q with some good asymptotic properties. We also give the asymptotic formula for the number of points on the stack Bun_G for asymptotically exact sequence of curves and a split reductive group G . In the case of GL_n we prove that the same formula holds if we take only semistable points in the account.

1. INTRODUCTION

The asymptotic formulas for various arithmetic invariants have always been of great interest to number theorists, starting from classical Brauer-Siegel theorem. Its natural generalisation is given by Tsfasman-Vlăduţ formula for the asymptotic behavior of class number of a curve. We give generalisations of this formula in two directions: the asymptotic formula for the quasi-residues of a family of l -adic sheaves having some good properties and asymptotic formula for the number of points on the stack of G -bundles on curve for reductive split G . For GL_n we are able to prove analogous formula for the number of semistable bundles.

2. GENERAL ASYMPTOTIC FORMULA FOR l -ADIC SHEAVES

Here we will describe the problem we are going to study and prove a general asymptotic formula for what we call *asymptotically exact system* with some additional global assumptions. By a system we mean nothing more than a collection of pairs (X_i, \mathcal{F}_i) , each pair consisting of a curve X_i and a constructible sheaf \mathcal{F}_i on it. We are interested in the behavior of quasi-residue of L -function of \mathcal{F}_i as i tends to infinity. The answer we are looking for should extract the result through some local invariants of the system, namely the Tsfasman-Vlăduţ invariants [TV]. The property of a system being asymptotically exact is exactly the property of these invariants to exist.

2.1. Asymptotical exactness. Let a *system* (Ξ, Φ) be a sequence of pairs (X_i, \mathcal{F}_i) , numbered by natural numbers, where X_i is some complete smooth curve over \mathbb{F}_q and $\mathcal{F}_i \in D_c^b(X_i, \mathbb{Q}_l)$ is an element of derived category of constructible l -adic sheaves on X_i . In all our applications all \mathcal{F}_i 's will be actual sheaves, but formulas can be proved for complexes without any loss of generality. For simplicity we will still call \mathcal{F}_i a sheaf even if it is actually an element of derived category.

Let $i : x \hookrightarrow X, x = \text{Spec}(\mathbb{F}_{q^n})$ be a closed point of X and let \mathcal{F} be a sheaf on X . Associated to this point we have a local L -factor

$$L_x(\mathcal{F}, s) = \frac{\prod_{i=2k} \det(1 - \text{Fr}_x q^{-ns} | H^i(i^* \mathcal{F}))}{\prod_{i=2k+1} \det(1 - \text{Fr}_x q^{-ns} | H^i(i^* \mathcal{F}))}.$$

That is actually the only local data of \mathcal{F}_i 's we are interested in.

Definition 2.2. *Tsfasman-Vlăduţ (TV) structure* Υ on a system (Ξ, Φ) is a collection of subsets $Z_{r,i} = Z_r(X_i) \subset |X_i|$, such that

- every $Z_{r,i}$ is finite,
- $|X_i| = \sqcup_r Z_{r,i}$,
- all the points in $Z_{r,i}$ are of the same degree,
- if $x_1 \in Z_{r,i}$ and $x_2 \in Z_{r,j}$, then there is an isomorphism $(\mathcal{F}_i)_{x_1} = (\mathcal{F}_j)_{x_2}$ as Frobenius-modules.

Morally TV-structure is just a particular choice of grouping points where local L -factors of corresponding sheaves are the same. Here is the most natural example of a system with a TV-structure:

Example 1: Let X_i be arbitrary and $\mathcal{F}_i = \underline{\mathbb{Q}}_l$ be a constant sheaf on X_i . Then grouping the points by degrees over \mathbb{F}_q defines a TV-structure $Z_{r,i} = \{x \in |X_i|, \deg x = r\}$. Further we will call this system $(\Xi, \underline{\mathbb{Q}}_l)$.

Example 2: Let X be a curve and \mathcal{F} be a local system on some open part $j : U \hookrightarrow X$. Let S denote the complement $X \setminus U$. Choose any local system \mathcal{L} on U and a sequence of ramified coverings $\varphi_i : X_i \rightarrow X$. By such data we can construct system $(\Xi, \Phi) = \{(X_i, \varphi_i^* j_! \mathcal{L})\}$. Here we can take also the full $\mathbb{R}^\bullet j_!$ or any of its sheaf cohomologies $\mathbb{R}^n j_!$; by our convention $j_!$ applied to an actual sheaf will usually mean the $\mathbb{R}^0 j_!$. Then by base change the stalks of \mathcal{F}_i are easy to describe: for a point $x \in |X_i|$ they depend only on the image $\varphi_i(x)$ and $\deg x$. So, taking by index set $|X| \times \mathbb{N}$, we can define a TV-structure in the following way: $Z_{y,r,i} = \{x \in |X_i|, \varphi_i(x) = y, \deg x = r\}$.

The next definition is made to distinguish systems with good asymptotic properties:

Definition 2.3. A system (Ξ, Φ) with TV-structure Υ is called *asymptotically exact*, if

- the genus $g_{X_i} \rightarrow \infty$ as $i \rightarrow \infty$
- for every r there exists limit

$$\gamma_r(\Xi, \Phi) = \lim_{i \rightarrow \infty} \frac{|Z_{r,i}|}{g_{X_i}}$$

We call these limits *Tsfasman-Vlăduţ invariants* of a system. They are direct generalisations of classical Tsfasman-Vlăduţ invariants in the case of $(\Xi, \underline{\mathbb{Q}}_l)$, γ_r are exactly β_r in [TV]. For index r , by $L(r)$ we will denote the value $L_x(F_i, 1)$ for any x in group of points indexed by r .

2.4. Quasi-residue of L -function. Let \mathcal{F} be a constructible l -adic sheaf on complete curve X . L -function $L(\mathcal{F}, s)$ is defined as a product

$$L(\mathcal{F}, s) = \prod_{x \in |X|} L_x(\mathcal{F}, s)^{-1} = \prod_{x \in |X|} \frac{\prod_{i=2k+1} \det(1 - \text{Fr}_x q^{-s \deg x} | H^i(i_x^* \mathcal{F}))}{\prod_{i=2k} \det(1 - \text{Fr}_x q^{-s \deg x} | H^i(i_x^* \mathcal{F}))},$$

where $i_x : x \hookrightarrow X$ is the embedding of the point x . By Weil conjectures L -function is a rational function of q^{-s} :

$$L(\mathcal{F}, s) = \frac{\prod_{i=2k+1} \det(1 - \text{Fr} \cdot q^{-s} | H^i(X, \mathcal{F}))}{\prod_{i=2k} \det(1 - \text{Fr} \cdot q^{-s} | H^i(X, \mathcal{F}))}.$$

In particular there is some integer r , such that there exist nonzero limit

$$\rho_{(X, \mathcal{F})} = \lim_{s \rightarrow 1} \frac{\mathcal{L}(\mathcal{F}, s)}{(s-1)^r},$$

which we call *quasi-residue* of \mathcal{F} .

For a constant sheaf we have $r = 1$ and

$$\rho_{(X, \underline{\mathbb{Q}}_l)} = \frac{q^{(1-g(X))} \cdot \#\text{Pic}^0(X)}{q-1}$$

2.5. Asymptotic formula for $\rho_{\mathcal{F}}$. Let (Ξ, Φ) be a system. We are interested in the asymptotical behavior of the quasi-residue as i grows, namely we want to find limit

$$\lim_{i \rightarrow \infty} \frac{\log_q \rho_{\mathcal{F}_i}}{g_{X_i}}.$$

In the case of $(\Xi, \underline{\mathbb{Q}}_l)$ which is asymptotically exact, the answer can be deduced from Tsfasman-Vlăduț formula [TV]

$$\lim_{i \rightarrow \infty} \frac{\log_q \rho_{X_i, \underline{\mathbb{Q}}_l}}{g_{X_i}} = - \sum_{m=1}^{\infty} \beta_m(\tilde{\mathcal{X}}) \log_q \left(\frac{q^m - 1}{q^m} \right).$$

Now we are going to prove a general formula for case of asymptotically exact system satisfying some good properties, listed below

Definition 2.6. We say that (Ξ, Φ) satisfy *local assumptions* if the following properties are satisfied

- there exist natural number n , such that for any r and $i > n$ all Frobenius-weights of stalks $(\mathcal{F}_i)_x$ are less or equal than $\frac{1}{2}$.
- the total dimension of the stalk $d((\mathcal{F}_i)_x) = \sum \dim H^i((\mathcal{F}_i)_x)$ is uniformly bounded by some number d (meaning $d((\mathcal{F}_i)_x) < d$ for any $x \in X_i$).

Remark 2.7. As we are interested only in asymptotic behavior, throwing out all (X_i, \mathcal{F}_i) with $i < n$ we get the system with the same asymptotic properties, so without loss of generality we can assume that condition from 2.6 is satisfied for all i

For $H^\bullet(X_i, \mathcal{F}_i)$ let $H_{<1}^\bullet(X_i, \mathcal{F}_i)$ denote the part of weight less than 1 with respect to Frobenius operator. Let also $H_{\geq 1}^\bullet(X_i, \mathcal{F}_i)$ be the transversal part of weight ≥ 1 .

Definition 2.8. We say that (Ξ, Φ) satisfy *global assumptions* if the following properties are satisfied

- $H^\bullet(X_i, \mathcal{F}_i)$ is a uniformly bounded graded space, namely there exist n , such that for any i $H^m(X_i, \mathcal{F}_i) > 0$, if $|m| > n$,
- the weights of $H^\bullet(X_i, \mathcal{F}_i)$ are uniformly bounded from above,
- the total dimension of $H_{\geq 1}^\bullet(X_i, \mathcal{F}_i)$ is $O(1)$,
- the total dimension of $H_{<1}^\bullet(X_i, \mathcal{F}_i)$ is $O(g_{X_i})$.
- there exist a weight $\omega < 1$, such that for any i all weights of $H_{<1}^\bullet(X_i, \mathcal{F}_i)$ are less than ω .

Theorem 2.9. (*General asymptotic formula*). Let (Ξ, Φ) be an asymptotically exact system with TV-structure Υ , satisfying local and global assumptions. Then we have the following formula:

$$\lim_{i \rightarrow \infty} \frac{\log_q \rho_{\mathcal{F}_i}}{g_{X_i}} = - \sum_r \gamma_r(\Xi, \Phi) \log_q L(r).$$

Remark 2.10. for $(\Xi, \underline{\mathbb{Q}}_l)$ we get classical Tsfasman-Vlăduț formula.

Proof: The proof follows essentially the original proof of Tsfasman and Vlăduț, but being adobted to the case of any sheaf.

Lemma 2.11. If (Ξ, Φ) with TV-structure Υ is asymptotically exact, then so does $(\Xi, \underline{\mathbb{Q}}_l)$

Proof: Just group $Z_{r,i}$ by the degree of points: let $B_m(X_i)$ be the set of points of degree m , then $B_m(X_i) = \cup_{r \in R'} Z_{r,i}$ for some R' and

$$\beta_m(\Xi) = \lim_{i \rightarrow \infty} \frac{|B_m(X_i)|}{g_{X_i}} = \sum_{r \in R'} \gamma(\Xi, \Phi).$$

The key point of the proof is Lefschetz fixed point formula: $\sum_i (-1)^i \text{Tr}(\text{Fr}^m q^{-m} | H^i(X, \mathcal{F})) = \sum_{x \in X(\mathbb{F}_{q^m})} \text{Tr}(\text{Fr}^m q^{-m} | \mathcal{F}_x)$. If a point $x \in X(\mathbb{F}_{q^m})$ actually came from x' defined over \mathbb{F}_{q^n} , then $\text{Tr}(\text{Fr}^m q^{-m} | \mathcal{F}_x) = \text{Tr}(\text{Fr}_{x'}^{\frac{m}{n}} q^{-m} | \mathcal{F}'_x)$ and there would be exactly n such summands (for all points got from $x' = \text{Spec } \mathbb{F}_{q^n}$ by base change to \mathbb{F}_{q^m}). So we can regroup the summands in Lefschetz formula in the following way:

$$\sum_j (-1)^j \text{Tr}(\text{Fr}^m q^{-m} | H^j(X_i, \mathcal{F})) = \sum_{n|m} n \sum_{Z_{r,i} \subset B_n} |Z_{r,i}| \text{Tr}(\text{Fr}_{x'}^{\frac{m}{n}} q^{-m} | \mathcal{F}_x, x \in Z_{r,i})$$

For simplicity we will denote $\text{Tr}(\text{Fr}_{x'}^{\frac{m}{n}} q^{-m} | \mathcal{F}_x, x \in Z_{r,i})$ by $\text{Tr}_{r,m}$. Let $f(g) : \mathbb{N} \rightarrow \mathbb{N}$ be a function such that $f(g) \rightarrow \infty$ as $g \rightarrow \infty$, and

$$\lim_{g \rightarrow \infty} \frac{f(g)}{\log g} = 0.$$

Let it in addition grow slow enough to satisfy the following property: for any $m < f(g)$ and any r , such that $Z_{r,i} \subset B_m(X_i)$, the difference

$$|\gamma_r(\Xi, \Phi) - \frac{|Z_r(X_i)|}{g_{X_i}}| \leq \frac{\epsilon(g_{X_i})}{|B_{\leq m}(X_i)|}$$

for some positive real-valued function $\epsilon(g) \rightarrow 0$ as $g \rightarrow \infty$. It is easy to see that such function f can be found easily for any countable set of converging sequences.

Lemma 2.12. *For such function $f(g)$*

$$\sum_{m=1}^{f(g_{X_i})} \frac{1}{m} \sum_{n|m} n \sum_{Z_{r,i} \subset B_{\leq n}} \text{Tr}_{r,m} \cdot |Z_r(X_i)| = \sum_{m=1}^{f(g_{X_i})} \sum_{Z_{r,i} \subset B_m} \log_q(L(r)) \cdot |Z_r(X_i)| + o(g_{X_i})$$

Proof: Let's change the summation order from the left side:

$$\sum_{m=1}^{f(g_{X_i})} \frac{1}{m} \sum_{n|m} n \sum_{Z_{r,i} \subset B_{\leq n}} \text{Tr}_{r,m} \cdot |Z_r(X_i)| = \sum_{n=1}^{f(g_{X_i})} \sum_{Z_{r,i} \subset B_{\leq n}} |Z_r(X_i)| \sum_{m=1}^{\lfloor \frac{f(g)}{n} \rfloor + 1} \frac{1}{m} \text{Tr}_{r,mn}.$$

Let's also expand the logarithm on the right side (which makes sense because of 2.6):

$$\sum_{m=1}^{f(g_{X_i})} \sum_{Z_{r,i} \subset B_{\leq n}} \log_q(L(r)) \cdot |Z_r(X_i)| = \sum_{m=1}^{f(g_{X_i})} \sum_{Z_{r,i} \subset B_m} |Z_r(X_i)| \sum_{n=1}^{\infty} \frac{1}{n} \text{Tr}_{r,nm}$$

We see that the difference is equal to

$$\sum_{m=1}^{f(g_{X_i})} \sum_{Z_{r,i} \subset B_m} |Z_r(X_i)| \sum_{n=\lfloor \frac{f(g)}{m} \rfloor}^{\infty} \frac{1}{m} \text{Tr}_{r,mn}.$$

Now, for any r , such that $Z_{r,i} \subset B_{\leq n}$, we have

$$|\text{Tr}_{r,nm}| = |\text{Tr}(\text{Fr}_x^{\frac{m}{n}} q^{-nm} | \mathcal{F}_x, x \in Z_{r,\bullet})| \leq d(\mathcal{F}_x) q^{-nm} < dq^{-\frac{nm}{2}}$$

using 2.6. So,

$$\begin{aligned}
\left| \sum_{m=1}^{f(gX_i)} \sum_{Z_{r,i} \subset B_m} |Z_r(X_i)| \sum_{n=\lfloor \frac{f(g)}{m} \rfloor + 1}^{\infty} \frac{1}{m} \text{Tr}_{r,mn} \right| &\leq d \sum_{m=1}^{f(gX_i)} \sum_{Z_{r,i} \subset B_m} |Z_r(X_i)| \sum_{n=\lfloor \frac{f(g)}{m} \rfloor + 1}^{\infty} q^{-\frac{nm}{2}} \leq \\
&\leq 3d \sum_{m=1}^{f(gX_i)} \sum_{Z_{r,i} \subset B_m} |Z_r(X_i)| q^{-\lfloor \frac{f(g)}{m} \rfloor m - m} = \\
&= 3d \sum_{m=1}^{f(gX_i)} |B_m(X_i)| q^{-\frac{1}{2} \lfloor \frac{f(g)}{m} \rfloor m - \frac{m}{2}} < \\
&< 3d \sum_{m=1}^{f(gX_i)} |B_m(X_i)| q^{-\frac{m}{2} - \frac{f(g)}{4}}
\end{aligned}$$

as $\lfloor \frac{f(g)}{m} \rfloor m \geq \frac{f(g)}{2}$ for $m \leq f(g)$.

In [TV] Tsfasman and Vlăduț have proven that for any $f(g)$, such that $\frac{f(g)}{\log g} \rightarrow 0$

$$\overline{\lim}_{i \rightarrow \infty} \frac{1}{g_{X_i}} \sum_{m=1}^{f(gX_i)} \frac{m B_m(X_i)}{q^{m/2} - 1} \leq 1,$$

so

$$\lim_{i \rightarrow \infty} q^{-\frac{f(g)}{4}} \frac{3d}{g_{X_i}} \sum_{m=1}^{f(gX_i)} B_m(X_i) q^{-\frac{m}{2}} \leq \lim_{i \rightarrow \infty} \frac{3d}{g_{X_i}} = 0.$$

Now we will prove that left and right sides of equation in 2.12 asymptotically coincide with the left and right sides of 2.9.

Lemma 2.13.

$$\lim_{i \rightarrow \infty} \frac{\log_q \rho_{\mathcal{F}_i}}{g_{X_i}} = \lim_{i \rightarrow \infty} \frac{1}{g_{X_i}} \sum_{m=1}^{f(gX_i)} \frac{1}{m} \sum_j (-1)^j \text{Tr}(\text{Fr}^m q^{-m} | H^j(X_i, \mathcal{F}_i)).$$

Proof: At first let's decompose $\rho_{\mathcal{F}_i}$ as a product $\rho_{\mathcal{F}_i}^+ \rho_{\mathcal{F}_i}^-$, where multiplicands correspond to $H_{\geq 1}^\bullet(X_i, \mathcal{F}_i)$ and $H_{< 1}^\bullet(X_i, \mathcal{F}_i)$. Then, as the total dimension of $H_{\geq 1}^\bullet(X_i, \mathcal{F}_i)$ is bounded and so are the weights, we conclude that $\lim_{i \rightarrow \infty} \frac{\log_q \rho_{\mathcal{F}_i}^+}{g_{X_i}} = 0$. So $\lim_{i \rightarrow \infty} \frac{\log_q \rho_{\mathcal{F}_i}}{g_{X_i}} = \lim_{i \rightarrow \infty} \frac{\log_q \rho_{\mathcal{F}_i}^-}{g_{X_i}}$.

In $\rho_{\mathcal{F}_i}^-$ all weights are < 1 , so we can expand the logarithm:

$$\log_q \rho_{\mathcal{F}_i}^- = \sum_{m=1}^{\infty} \frac{1}{m} \sum_j (-1)^j \text{Tr}(\text{Fr}^m q^{-m} | H_{\leq 1}^j(X_i, \mathcal{F}_i)).$$

Now it is left to prove two facts: that

$$\lim_{i \rightarrow \infty} \frac{1}{g_{X_i}} \sum_{m=f(g)+1}^{\infty} \frac{1}{m} \sum_j (-1)^j \text{Tr}(\text{Fr}^m q^{-m} | H_{< 1}^j(X_i, \mathcal{F}_i)) = 0$$

and that

$$\lim_{i \rightarrow \infty} \frac{1}{g_{X_i}} \sum_{m=1}^{f(g)} \frac{1}{m} \sum_j (-1)^j \text{Tr}(\text{Fr}^m q^{-m} | H_{\geq 1}^j(X_i, \mathcal{F}_i)) = 0$$

Expression in the first one can be bounded by geometric progression and the total dimension of weight < 1 part:

$$\lim_{i \rightarrow \infty} \frac{1}{g_{X_i}} \sum_{f(g)+1}^{\infty} \frac{1}{m} \sum_j (-1)^j \text{Tr}(\text{Fr}^m q^{-m} | H_{<1}^j(X_i, \mathcal{F}_i)) \leq \lim_{i \rightarrow \infty} \frac{d(H_{<1}^\bullet(X_i, \mathcal{F}_i))}{g_{X_i}} \frac{q^{-(1-\omega)f(g_{X_i})}}{q^{(1-\omega)} - 1} = 0$$

as $f(g_{X_i}) \rightarrow \infty$ and $d(H_{<1}^\bullet(X_i, \mathcal{F}_i))$ is $O(g_{X_i})$.

The second follows just from the uniform boundness of weights and the fact that $d(H_{\geq 1}^\bullet(X_i, \mathcal{F}_i))$ is $O(1)$: let α be the bound for weights, then

$$\begin{aligned} \lim_{i \rightarrow \infty} \frac{1}{g_{X_i}} \sum_{m=1}^{f(g)} \frac{1}{m} \sum_j (-1)^j \text{Tr}(\text{Fr}^m q^{-m} | H_{\geq 1}^j(X_i, \mathcal{F}_i)) &\leq \lim_{i \rightarrow \infty} \frac{d(H_{\geq 1}^\bullet(X_i, \mathcal{F}_i))}{g_{X_i}} \sum_{m=1}^{f(g)} \frac{1}{m} q^{(\alpha-1)m} \leq \\ &\leq \lim_{i \rightarrow \infty} \frac{d(H_{\geq 1}^\bullet(X_i, \mathcal{F}_i)) q^{(\alpha-1)(f(g)+1)}}{g_{X_i}} = \\ &= 0, \end{aligned}$$

as $\lim_{g \rightarrow \infty} \frac{f(g)}{\log g} = 0$. The statement of the lemma follows.

The final step of the proof is given by this last lemma

Lemma 2.14.

$$\lim_{i \rightarrow \infty} \frac{1}{g_{X_i}} \sum_{m=1}^{f(g_{X_i})} \sum_{Z_{r,i} \subset B_m} \log_q(L(r)) \cdot |Z_r(X_i)| = \sum_r \gamma_r(\Xi, \Phi) \log_q L(r).$$

Proof: Again due to 2.6 we know that for $Z_r(X_i) \subset B_m(X_i)$

$$\log_q L(r) \leq d \log_q(1 - q^{-\frac{m}{2}}) \leq 3dq^{-\frac{m}{2}}$$

Remembering the properties of $f(g)$ we get

$$\begin{aligned} &\left| \frac{1}{g_{X_i}} \sum_{m=1}^{f(g_{X_i})} \sum_{Z_{r,i} \subset B_m} \log_q(L(r)) \cdot |Z_r(X_i)| - \sum_r \gamma_r(\Xi, \Phi) \log_q L(r) \right| \leq \\ &\leq 3\epsilon(g_{X_i})d \sum_{m=1}^{f(g_{X_i})} q^{-\frac{m}{2}} \rightarrow 0, \text{ as } i \rightarrow \infty \end{aligned}$$

This ends the proof of the lemma.

It's easy to see that three proven lemmas (2.12, 2.13, 2.14) give the statement of the theorem.

3. ASYMPTOTIC FORMULAS FOR G -BUNDLES

Here we will apply the results of previous section to find the asymptotic behaviour of the number of points on stacks Bun_G for G split. Using Zagier's formula for the number of semistable bundles (for GL_n) we also prove that the answer does not change if we restrict ourselves only to semistable part of Bun_{GL} .

3.1. Asymptotic formula for $|\text{Bun}_G^0(\mathbb{F}_q)|$. For split reductive group G , Bun_G as usual denotes the stack of G -bundles on curve X , Bun_G^0 denotes the connected component of the trivial bundle. By $|\text{Bun}_G^0(\mathbb{F}_q)|$ we mean the number of its points in the stacky sense, namely

$$|\text{Bun}_G^0(\mathbb{F}_q)| = \sum_{x \in \text{Bun}_G^0(\mathbb{F}_q)} \frac{1}{|\text{Aut} x|}.$$

We call it the G -mass of X and further denote by $M_G(X)$.

To introduce the formula for $M_G(X)$, we need at first to express it through the quasi-residues of some L -functions.

Let G be a reductive group over \mathbb{F}_q of dimension d , with Borel subgroup $B \subset G$, defined over \mathbb{F}_q , with maximal torus $T \subset B$ also defined over \mathbb{F}_q . Let $V = S^\bullet(X(T) \otimes \mathbb{Q})$ and $W = (N_G(T)/T)(\overline{\mathbb{F}}_q)$. Then $V = S^\bullet(X(T) \otimes \mathbb{Q})^W$ is an algebra of polynomials $S^\bullet(V_1 \oplus \dots \oplus V_{\dim G/B})$ and each V_i is a module over $\text{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$. Then we have Steinberg's formula:

$$|G(\mathbb{F}_q)| = q^d \prod_{i \geq 1} \det(1 - q^{-i} \text{Fr} | V_i).$$

Let also \widehat{G} be the group of characters $\widehat{G} \hookrightarrow X(T)$. Then we have $V_1 \cong \widehat{G}$. Each of V_i tautologically defines a sheaf on $\text{Spec } \mathbb{F}_q$, and by pull-back also a sheaf $\mathcal{V}_i = \text{pr}_X^* V_i$ on X .

Let G be a reductive group over $\mathbb{F}_q(X)$. Choose any left invariant differential top form ω on G . Then ω defines top-forms ω_x on $G(F_q(X)_{\hat{x}})$ for every closed point $x \in X$, while each ω_x defines left Haar measure on $G(F_q(X)_{\hat{x}})$. Tamagawa measure ω_G on its adelic points $G(\mathbb{A}_X)$ is defined as $q^{(1-g_X)d} \prod_{x \in X} L_x(\mathcal{V}_1, 1) \omega_x$ and does not depend on the choice of ω . Tamagawa number τ_G is defined as $\rho_{\widehat{G}}^{-1} \omega_G(G(K) \backslash G^1(\mathbb{A}))$. Quasi-discriminant D_G denotes $\omega_G(G^c(\mathbb{A}_X))^{-2}$, where $G^c(\mathbb{A}_X) \subset G(\mathbb{A}_X)$ is the maximal compact subgroup.

It is easy to see that for G split and ω defined over \mathbb{F}_q we have

$$\omega_x(G(\mathcal{O}_x)) = \frac{|G(\mathbb{F}_{q^{\deg x}})|}{q^{d \deg x}} = \prod_{i \geq 1} \det(1 - q^{-i \deg x} \text{Fr}_x | V_i) = \prod_{i \geq 1} L_x(\mathcal{V}_i, i) = \prod_{i \geq 1} L_x(\mathcal{V}_i(1-i), 1)$$

, so, as $G^c(\mathbb{A}_X) = \prod_{x \in X} G(\mathcal{O}_x)$ we get that

$$\omega_G(G^c(\mathbb{A}_X)) = \prod_{i \geq 2} L(\mathcal{V}_i(1-i), 1)^{-1} = \prod_{i \geq 2} \rho_{\mathcal{V}_i(1-i)}^{-1}$$

The Siegel's mass formula (see [HN]) says that

$$M_G(X) = \rho_{\widehat{G}} \tau_G D_G^{1/2} = \tau_G q^{g_X - 1} \dim G \prod_{i \geq 1} \rho_{\mathcal{V}_i(1-i)}$$

Let's now take some asymptotically exact system $(\Xi, \underline{\mathbb{Q}}_l)$. What is the asymptotic of $M_G(X_i)$ as i grows? Applying the result of the previous section, we can easily give the answer.

Theorem 3.2. *Let $(\Xi, \underline{\mathbb{Q}}_l)$ be asymptotically exact and let G be a split reductive group over \mathbb{F}_q . Then*

$$\lim_{i \rightarrow \infty} \frac{\log_q M_G(X_i)}{g_{X_i}} = \dim G - \sum_{r=1}^{\infty} \gamma_r(\Xi, \underline{\mathbb{Q}}_l) \log_q \left(\frac{|G(\mathbb{F}_{q^r})|}{q^{r \dim G}} \right)$$

Remark 3.3. for $G = \mathbb{G}_m$, $\text{Bun}_G^0 = \text{Pic}^0(X)$ and we get classical Tsfasman-Vlăduț formula:

$$\lim_{i \rightarrow \infty} \frac{\log_q |\text{Pic}^0(X_i)|}{g_{X_i}} = 1 - \sum_{r=1}^{\infty} \gamma_r(\Xi, \underline{\mathbb{Q}}_l) \log_q \left(\frac{q^r - 1}{q^r} \right)$$

Proof:

Applying \log_q to Siegel formula we get

$$\begin{aligned}
\lim_{i \rightarrow \infty} \frac{\log_q M_G(X_i)}{g_{X_i}} &= \lim_{i \rightarrow \infty} \frac{\log_q \tau_G}{g_{X_i}} + \lim_{i \rightarrow \infty} \frac{(g_{X_i} - 1) \dim G}{g_{X_i}} + \lim_{i \rightarrow \infty} \frac{\log_q \prod_{i \geq 1} \rho_{V_i(i-1)}}{g_{X_i}} = \\
&= \lim_{i \rightarrow \infty} \frac{\log_q \tau_G}{g_{X_i}} + \dim G - \sum_{r=1}^{\infty} \gamma_r(\Xi, \underline{\mathbb{Q}}_l) \log_q \left(\frac{|G(\mathbb{F}_{q^r})|}{q^{r \dim G}} \right),
\end{aligned}$$

where the last equality is due to Steinberg's formula. Now, assuming Weil conjectures for semisimple groups over function fields (claimed to be proved by Lurie and Gaitsgory, see [GL] for the first written down part of the proof) we can use [[BD], Corollary 6.8.], which says that provided G split, τ_G is constant (does not depend on X_i). So $\lim_{i \rightarrow \infty} \frac{\log_q \tau_G}{g_{X_i}} = 0$ and we get the desired formula.

3.4. Asymptotic formula for stable bundles. Here we restrict ourselves to the case $G = \mathrm{GL}_n$. We are going to extend the formula for $M_{\mathrm{GL}_n}(X)$ to its semistable part $M_{\mathrm{GL}_n}^{ss}(X)$. Now we do not have anything like Siegel mass formula, but it comes out that $M_{\mathrm{GL}_n}^{ss}(X)$ can be expressed through $M_{\mathrm{GL}_m}(X)$ for $m \leq n$ (this is the formula of Don Zagier, see [[M], Theorem 5.3] or [[Z], Theorem 3]):

$$M_{\mathrm{GL}_n}^{ss}(X) = \sum_{n_1 + \dots + n_k = n} q^{(g_X - 1) \sum_{i < j} n_i n_j} \prod_{i=1}^{k-1} \frac{q^{(n_i + n_{i+1}) \{(n_1 + \dots + n_i) n / d\}}}{1 - q^{n_i + n_{i+1}}} M_{\mathrm{GL}_{n_1}}(X) \cdots M_{\mathrm{GL}_{n_k}}(X)$$

We need to show that the first summand with $n_1 = n, n_i = 0$ is asymptotically the biggest.

$$\begin{aligned}
(1) \quad \lim_{i \rightarrow \infty} \frac{\log_q M_{\mathrm{GL}_n}(X_i)}{g_{X_i}} &= n^2 - \sum_{r=1}^{\infty} \gamma_r(\Xi, \underline{\mathbb{Q}}_l) \log_q \left(\frac{\prod_{i=1}^n (q^{rn} - q^{r(n-i)})}{q^{rn^2}} \right) = \\
(2) \quad &= n^2 - \sum_{r=1}^{\infty} \gamma_r(\Xi, \underline{\mathbb{Q}}_l) \sum_{i=1}^n \log_q (1 - q^{-ir}).
\end{aligned}$$

Now, bounding $-\log_q(1 - q^{-ir})$ by geometric progression, we get that $\log_q(1 - q^{-ir}) \leq q^{-ir+1}$, so again using Tsfasman-Vlăduţ bound [TV]

$$\lim_{i \rightarrow \infty} \frac{\log_q M_{\mathrm{GL}_n}(X_i)}{g_{X_i}} \leq n^2 - \sum_{r=1}^{\infty} \gamma_r(\Xi, \underline{\mathbb{Q}}_l) n q^{-ir+1} < n^2 - n.$$

Now, let's look at the limit of each summand: for n_1, n_2, \dots, n_k we get

$$\sum_{i < j} n_i n_j + \sum_{j=1}^k \lim_{i \rightarrow \infty} \frac{\log_q M_{\mathrm{GL}_{n_j}}(X_i)}{g_{X_i}} < \sum_{i < j} n_i n_j + \sum_{j=1}^k n_j^2 + n_j$$

But as $n = \sum n_i$, $n^2 = (\sum n_i)^2 \geq \sum_{i < j} n_i n_j + \sum_{j=1}^k n_j^2 + n_j$. In logarithm term for $n_1 = n$ we have $n^2 + \text{something positive}$, because $|\mathrm{GL}_n(\mathbb{F}_q)| < q^{n^2}$, so it grows faster than any other, so

$$\lim_{i \rightarrow \infty} \frac{\log_q M_{\mathrm{GL}_n}(X_i)}{g_{X_i}} = \lim_{i \rightarrow \infty} \frac{\log_q M_{\mathrm{GL}_n}^{ss}(X_i)}{g_{X_i}}$$

Let $\mathcal{M}_{\mathrm{GL}_n(X)}^0$ denote the moduli space of semistable vector bundles on X , whose determinant bundle is trivial. Now, as there are only finitely many semistable bundles and the order of

automorphism group of semistable bundle is bounded by the order of $|\mathrm{GL}_n(\mathbb{F}_q)|$, the asymptotic of $M_{\mathrm{GL}_n}^{ss}(X_i)$ is the same as for $|\mathcal{M}_{\mathrm{GL}_n(X)}^0(\mathbb{F}_q)|$. So we get the following theorem

Theorem 3.5. *Let $(\Xi, \underline{\mathbb{Q}}_l)$ be asymptotically exact. Then*

$$\lim_{i \rightarrow \infty} \frac{\log_q |\mathcal{M}_{\mathrm{GL}_n(X)}^{ss}(\mathbb{F}_q)|}{g_{X_i}} = \dim G - \sum_{r=1}^{\infty} \gamma_r(\Xi, \underline{\mathbb{Q}}_l) \log_q \left(\frac{|G(\mathbb{F}_{q^r})|}{q^{r \dim G}} \right)$$

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